

August 2, 2012

# Complex Factorisation and Recursion for One-Loop Amplitudes

Sam D. Alston, David C. Dunbar and Warren B. Perkins

*College of Science,  
Swansea University,  
Swansea, SA2 8PP, UK  
August 2, 2012*

## Abstract

We consider the factorisation of one-loop amplitudes at complex kinematic points. By determining the terms that are absent for real kinematics, we can construct a recursive ansatz for the purely rational pieces of one-loop amplitudes in massless theories. We illustrate this method by verifying the Bern *et.al.*  $n$ -point ansatz for the *single-minus* one-loop amplitudes in Yang-Mills theory and by constructing the scalar contribution to the one-loop five graviton MHV scattering amplitude.

PACS numbers: 04.65.+e

## I. INTRODUCTION

In recent years significant progress has been made in the computation of scattering amplitudes in gauge theories and gravity by utilizing the analytic properties of these amplitudes [1–5]. One ingredient in this process has been the use of factorisation properties when the momenta in the amplitude have been complexified<sup>1</sup>. In particular, on-shell recursive methods have been very useful in the evaluation of many massless tree level processes [4, 5]. For example, consider shifting two of the external momenta according to:

$$\begin{aligned}\bar{\lambda}_i &\longrightarrow \hat{\bar{\lambda}}_i = \bar{\lambda}_i - z\bar{\lambda}_j \\ \lambda_j &\longrightarrow \hat{\lambda}_j = \lambda_j + z\lambda_i\end{aligned}\tag{1.1}$$

where  $z$  is a complex parameter. Providing the shifted amplitude  $A(z)$  is analytic and vanishes at large  $|z|$ , then by Cauchy's theorem, we may obtain the unshifted function from the residues at the poles in  $A(z)$ ,

$$A(0) = - \sum_i \text{Res} \left( \frac{A(z)}{z} \right) \Big|_{z_i}\tag{1.2}$$

This is only useful if we can evaluate the residues and hence an understanding of the singularity structure of the amplitude is essential. At tree level the factorisation is relatively simple: amplitudes must factorise on multi-particle and collinear poles. Defining  $K^\mu \equiv \sum_{j=i}^{i+r-1} k_j^\mu$ , as  $K$  becomes null the  $n$ -point tree amplitude  $A_n^{\text{tree}}$  factorises as

$$A_n^{\text{tree}} \xrightarrow{K^2 \rightarrow 0} \sum_\lambda \left[ A_{r+1}^{\text{tree}}(k_i, \dots, k_{i+r-1}, K^\lambda) \frac{i}{K^2} A_{n-r+1}^{\text{tree}}(-K^{-\lambda}, k_{i+r}, \dots, k_{i-1}) \right]\tag{1.3}$$

where  $\lambda$  denotes the helicity of the intermediate state. Consequently, simple poles in the shifted amplitude  $A(z)$  occur at values of  $z$  where  $K^2(z) = 0$ . Since  $k_a + k_b$  is independent of  $z$ , only those  $K$ 's containing precisely one of  $k_a$  or  $k_b$  will be  $z$  dependent. When the corresponding  $K^2(z)$  vanishes the residue will be the product of the tree amplitudes defined at  $z = z_i$ . Thus the  $n$ -point tree amplitude can be expressed in terms of lower point amplitudes:

$$A_n^{\text{tree}}(0) = \sum_{i,\lambda} A_{r+1}^{\text{tree},\lambda}(z_i) \frac{i}{K^2} A_{n-r+1}^{\text{tree},-\lambda}(z_i),\tag{1.4}$$

where the summation over  $i$  is only over factorisations where the  $a$  and  $b$  legs are on opposite sides of the pole. This is the on-shell recursive expression of [4].

There are several complications in applying these techniques beyond tree level. Firstly, loop amplitudes can develop higher order singularities for complex momenta. While these do not block recursion *per se*, they do necessitate an understanding of factorisation beyond leading order. Suppose a rational function has a double pole so that

$$R(z) = \frac{\alpha}{(z - z_i)^2} + \frac{\beta}{(z - z_i)} + \dots\tag{1.5}$$

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<sup>1</sup> A null momentum can be represented as a pair of two component spinors  $p^\mu = \sigma_{\alpha\dot{\alpha}}^\mu \lambda^\alpha \bar{\lambda}^{\dot{\alpha}}$ . For real momenta  $\lambda = \pm \bar{\lambda}^*$  but for complex momenta  $\lambda$  and  $\bar{\lambda}$  are independent [6].

then

$$\text{Res}\left(\frac{R(z)}{z}\right)\bigg|_{z_i} = -\frac{\alpha}{z_i^2} + \frac{\beta}{z_i} \quad (1.6)$$

and both the leading and sub-leading, or ‘pole under the double pole’, terms are needed in order to apply recursion as in eq. (1.2). In general, the structure of the sub-leading poles is poorly understood.

Secondly, in general loop amplitudes contain both rational and non-rational pieces. One strategy for computing one-loop amplitudes is to split the amplitude into a *cut-constructible* piece and a purely rational piece,

$$A_n = C_n + R_n \quad (1.7)$$

The  $C_n$  may be computed using unitarity techniques [2, 3, 7–10] and the remaining  $R_n$  may then, in principle, be determined recursively via Cauchy’s theorem *provided* the singularities of  $R_n = A_n - C_n$  are understood.

A general  $n$ -point one-loop amplitude in a massless theory such as gravity or QCD can be expanded in terms of loop momentum integrals,  $I_m[P^d(\ell)]$ , where  $m$  denotes the number of vertices in the loop and  $P^d(\ell)$  is a polynomial of degree  $d$  in the loop momentum  $\ell$ . Performing a Passarino-Veltman reduction [11] on the loop momentum integrals yields an amplitude (to  $O(\epsilon)$  in the dimensional reduction parameter  $\epsilon$ ),

$$A_n^{1\text{-loop}} = \sum_i c_i I_4^i + \sum_j d_j I_3^j + \sum_k e_k I_2^k + R_n, \quad (1.8)$$

where  $c_i, d_j, e_k$  and  $R_n$  are rational functions and the  $I_4, I_3$ , and  $I_2$  are scalar box, triangle and bubble functions respectively. The mathematical form of these integral functions depends on whether the momenta flowing into a vertex are null (massless) or not (massive) [12]. In terms of this basis we can define

$$C_n = \sum_i c_i I_4^i + \sum_j d_j I_3^j + \sum_k e_k I_2^k \quad (1.9)$$

The coefficients,  $c_i, d_j$  and  $e_k$ , contain a range of singularities that are not present in the full amplitude. Individual coefficients may contain *spurious singularities* of the form  $\Delta^{-P}$ , where  $\Delta$  is a Gram-determinant of an integral function, yet the entire amplitude is finite as  $\Delta \rightarrow 0$ . These singularities in the coefficients can be of high-order in  $\Delta$ : we will encounter a case of  $P = 5$  in one of our examples. These spurious singularities cancel amongst the terms in  $C_n$  and also, crucially, with the rational term  $R_n$ . There are also singularities that occur at the same kinematic points as the physical singularities, but are of higher order. Again cancellations between the terms in  $C_n$  and  $R_n$  must remove these higher order poles from the complete amplitude.

Starting from  $C_n$ , we can view this cancellation constraint as a means of generating parts of  $R_n$  as higher order poles generate rational *descendants* from the terms in  $C_n$ . To evaluate the residue at a higher order pole the integral functions must be expanded to a corresponding order and the derivatives in this Taylor expansion eventually yield rational terms. In this way we obtain rational descendant terms whose origins lie in both the box and bubble integral contributions to  $C_n$ . This does not completely specify  $R_n$  but leaves the unspecified component free of these higher order singularities.

In the following sections we describe how, by using axial gauge methods to understand the complex factorisation, we can apply recursion to the rational parts of one-loop

amplitudes. Specifically we consider two examples: the  $n$ -point single-minus one-loop Yang-Mills amplitude  $A_n(1^-, 2^+, \dots, n^+)$  and the five-point scalar supergravity amplitude  $M_5(1^-, 2^-, 3^+, 4^+, 5^+)$ . The  $n$ -point amplitude  $A_n(1^-, 2^+, \dots, n^+)$  vanishes at tree level and consequently is a purely rational one-loop amplitude. As such it has no cut constructible parts but it does have multiple poles in complex momentum. This amplitude was originally computed using off-shell methods [13]. In ref. [14] a form for the sub-leading singularity was postulated and recursion used to (re)obtain the  $n$ -point formulae. Here we will prove, using axial gauge methods, the explicit form of the sub-leading term for the shift used in ref. [14].

The second example is a case where rational descendants of the integral functions in  $C_n$  contribute to the rational terms. The example is that of one-loop five graviton scattering where we have a scalar circulating in the loop. This is the last remaining calculation to complete the five-graviton scattering amplitude. The expansion of this amplitude in the form of (1.8) is plagued by high order singularities. We obtain  $R_5$  recursively and describe how these high order singularities generate contributions to  $R_5$  in addition to those coming directly from standard and non-standard factorisations.

## II. COMPLEX FACTORISATION

For real momenta the factorisation of one-loop massless amplitudes is described in ref. [15],

$$\begin{aligned} A_n^{1\text{-loop}} \xrightarrow{K^2 \rightarrow 0} & \sum_{\lambda=\pm} \left[ A_{r+1}^{1\text{-loop}}(k_i, \dots, k_{i+r-1}, K^\lambda) \frac{i}{K^2} A_{n-r+1}^{\text{tree}}((-K)^{-\lambda}, k_{i+r}, \dots, k_{i-1}) \right. \\ & + A_{r+1}^{\text{tree}}(k_i, \dots, k_{i+r-1}, K^\lambda) \frac{i}{K^2} A_{n-r+1}^{1\text{-loop}}((-K)^{-\lambda}, k_{i+r}, \dots, k_{i-1}) \\ & \left. + A_{r+1}^{\text{tree}}(k_i, \dots, k_{i+r-1}, K^\lambda) \frac{i}{K^2} A_{n-r+1}^{\text{tree}}((-K)^{-\lambda}, k_{i+r}, \dots, k_{i-1}) F_n(K^2; k_1, \dots, k_n) \right], \end{aligned} \quad (2.1)$$

where the one-loop ‘factorisation function’  $F_n$  is helicity-independent. This factorisation has single poles in  $K^2$ . We refer to the singularities given in this equation as the standard factorisations. Singularities not contained in eq. (2.1) we refer to as “non-standard” factorisations.

For complex momenta we can acquire higher order poles. For a two-particle pole  $K^2 = (k_a + k_b)^2 = 2k_a \cdot k_b = \langle a b \rangle [b a]$ . For real momentum  $\langle a b \rangle = \pm [a b]^*$  and so both vanish at the pole<sup>2</sup>. However for complex momenta we may have  $\langle a b \rangle = 0$  but  $[a b] \neq 0$ . So terms such as  $[a b]^2 / \langle a b \rangle^2$  which are finite for real momenta can have multiple poles for complex momenta. These can be interpreted within eq. (2.1) as arising from the three-point one-loop amplitude acquiring a singularity. Specifically, the three-point all-plus (or all-minus) one-loop amplitude has a pole [14]

$$A_3^{1\text{-loop}}(K^+, a^+, b^+) = \frac{1}{K^2} V^{1\text{-loop}}(K^+, a^+, b^+) \quad (2.2)$$

<sup>2</sup> As usual we are using a spinor helicity formalism with the usual spinor products  $\langle j l \rangle \equiv \langle j^- | l^+ \rangle = \bar{u}_-(k_j) u_+(k_l)$  and  $[j l] \equiv \langle j^+ | l^- \rangle = \bar{u}_+(k_j) u_-(k_l)$ . In terms of spinors  $\langle a b \rangle = \epsilon_{\alpha\beta} \lambda_a^\alpha \lambda_b^\beta$  and  $[a b] = -\epsilon_{\dot{\alpha}\dot{\beta}} \bar{\lambda}_a^{\dot{\alpha}} \bar{\lambda}_b^{\dot{\beta}}$ . We also use  $[i | K_{abc} | j \rangle$  to denote  $\langle i^+ | K_{abc} | j^+ \rangle$  with  $K_{abc}^\mu = k_a^\mu + k_b^\mu + k_c^\mu$  etc. Also  $s_{ab} = (k_a + k_b)^2$ ,  $t_{abc} = (k_a + k_b + k_c)^2$ , etc.

where, for pure Yang–Mills,

$$V^{1\text{-loop}}(K^+, a^+, b^+) = -\frac{i}{48\pi^2} [K a] [a b] [b K]. \quad (2.3)$$

For real momenta  $A_3^{1\text{-loop}}(K^+, a^+, b^+)$  vanishes as  $K^2 \rightarrow 0$  but it can be singular for complex momenta. Equation (2.2) specifies the double pole as  $K^2 \rightarrow 0$  however, as discussed previously, we require the subleading pole in order to use recursion.

As an example of the structure of the double pole consider the amplitude with a single minus helicity  $A_n(a^-, b^+, \dots, n^+)^3$ . This amplitude vanishes to all orders in perturbation theory in a supersymmetric theory and consequently at tree level in Yang–Mills. It is non-vanishing at one-loop level but, since the tree amplitude vanishes, is entirely rational. The all- $n$  form was first obtained by Mahlon [13] using off-shell recursion [16]. In [14] the complex factorisation of the single-minus one-loop Yang–Mills amplitude was considered by applying the shift of eq. (1.1) to the  $\bar{\lambda}$  of the negative helicity leg and the  $\lambda$  of an adjacent positive helicity leg. For this specific case a form for the ‘pole under the double pole’ was proposed. Using this and applying complex recursion the following form for the amplitude was presented:

$$\begin{aligned} A_n^{1\text{-loop}}(a^-, b^+, \dots, n^+) &= A_{n-1}^{1\text{-loop}}(d^+, \dots, n^+, \hat{a}^-, \hat{K}_{bc}^+) \frac{i}{K_{bc}^2} A_3^{\text{tree}}(\hat{b}^+, c^+, -\hat{K}_{bc}^-) \\ &+ \sum_{i=4}^{n-1} A_{n-i+2}^{\text{tree}}((i+1)^+, \dots, n^+, \hat{a}^-, \hat{K}_{b\dots i}^-) \frac{i}{K_{b\dots i}^2} A_i^{1\text{-loop}}(\hat{b}^+, \dots, i^+, -\hat{K}_{2\dots i}^+) \\ &+ A_{n-1}^{\text{tree}}(d^+, \dots, n^+, \hat{a}^-, \hat{K}_{bc}^-) \frac{i}{(K_{bc}^2)^2} V^{1\text{-loop}}(\hat{b}^+, c^+, -\hat{K}_{bc}^+) \\ &\times \left( 1 + K_{bc}^2 \mathcal{S}^{(0)}(\hat{a}, \hat{K}_{bc}^+, d) \mathcal{S}^{(0)}(c, -\hat{K}_{bc}^-, \hat{b}) \right), \end{aligned} \quad (2.4)$$

where

$$\mathcal{S}^{(0)}(a, s^+, b) = \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle} \quad \text{and} \quad \mathcal{S}^{(0)}(a, s^-, b) = -\frac{[ab]}{[as][sb]}. \quad (2.5)$$

Expression (2.4) was shown to match that of Mahlon [13] up to  $n = 15$ . This expression has also been justified using gauge Lorentz invariance [17]. The form of the subleading pole used to generate (2.4) in terms of soft-factors is only valid for the particular shift used [18]. In the next section we provide an explicit constructive derivation of the sub-leading terms based on a diagrammatic analysis using axial gauge rules.

### III. $n$ -POINT SINGLE MINUS YANG MILLS AMPLITUDES

In this section we study the factorisation of the single-minus Yang–Mills amplitudes  $A(a^-, b^+, c^+, \dots, n^+)$  under a shift of legs  $a$  and  $b$  as above. The diagrams in fig. 1 generate the standard factorisations given in eq.(2.1). As  $\langle \hat{bc} \rangle \rightarrow 0$  double poles come from diagrams of the form illustrated in fig. 2 where the current  $\tau_n$  is the sum of all possible sub-diagrams. To evaluate these diagrams we use axial-gauge rules [19]. In this scheme internal off-shell particles are still labelled by  $\pm$  helicity and the non-vanishing three-point vertices

<sup>3</sup> As usual we are considering the colour-ordered partial amplitudes.

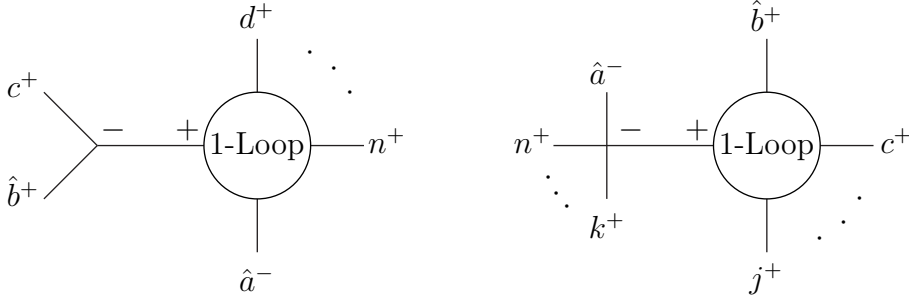


FIG. 1: Standard Factorisations of  $A(\hat{a}^-, \hat{b}^+, c^+, \dots, n^+)$ .

are the MHV and  $\overline{\text{MHV}}$  vertices

$$V_3(1^-, 2^-, 3^+) = i \frac{\langle 12 \rangle [3q]^2}{[1q][2q]} \quad V_3(1^+, 2^+, 3^-) = i \frac{[21] \langle 3q \rangle^2}{\langle 1q \rangle \langle 2q \rangle} \quad (3.1)$$

where  $q$  is a reference null vector. For non-null momenta,  $P$ , we define

$$|P\rangle \equiv P|q\rangle, \quad |P] \equiv -\frac{P|q\rangle}{2P \cdot q} \quad (3.2)$$

which corresponds to using  $q$ -nullified momenta  $P^b$ , where

$$P^b \equiv P - \frac{P^2}{2P \cdot q} q. \quad (3.3)$$

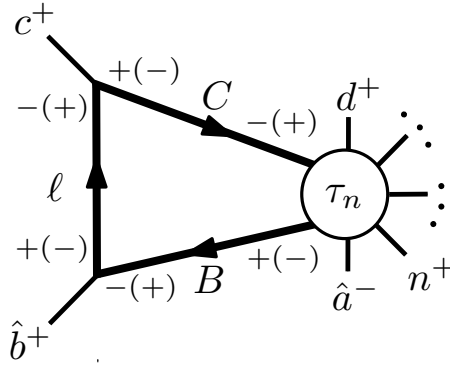


FIG. 2: The set of diagrams corresponding to the non-standard factorisation as  $\langle bc \rangle \rightarrow 0$  can be gathered together as shown.  $\tau_n$  is the set of diagrams with all but two legs on-shell.

We can simplify this particular computation considerably by setting  $\lambda_q = \lambda_a$  (and leaving  $\bar{\lambda}_q$  arbitrary at this stage). With this choice three-point  $\overline{\text{MHV}}$  vertices and four-point MHV vertices involving  $a^-$  vanish. Hence the  $a^-$  leg must be attached to a three-point MHV vertex and each diagram contains a single three-point MHV vertex and  $n - 1$  three-point  $\overline{\text{MHV}}$  vertices<sup>4</sup>.

<sup>4</sup> Consider constructing a one-loop diagram with  $n_-$  negative helicity external legs and  $n_+$  positive helicity

Since the negative helicity leg  $a^-$  must be attached to the only MHV three-point vertex, the diagrams contributing to the non-standard  $\langle bc \rangle$  poles have the helicity structure shown in fig. 2. In this diagram  $\tau_n$  is a current, or off-shell tree amplitude. We label the off-shell legs (which depend upon loop-momenta)

$$B = \ell + k_b \quad \text{and} \quad C = \ell - k_c. \quad (3.4)$$

Note that  $B - C = k_b + k_c$ . As we will see, one  $\langle bc \rangle^{-1}$  factor arises from the tree current  $\tau_n$  and a second from the loop integration, specifically the region where  $\ell$ ,  $B$  and  $C$  are all close to null. Throughout, we view the unshifted amplitude as a sum of functions, each of which corresponds to a Feynman diagram involving real momenta. In particular the loop momenta are real and where we indicate on a diagram which legs will ultimately be shifted, that shift applies to the function obtained by evaluating the diagram with real momenta.

The contribution from fig. 2 is then

$$C^{\text{n-s f}} = \int d^d \ell \frac{[b|\ell|a][c|\ell|a]}{\langle ba \rangle \langle ca \rangle} \frac{\langle Ca \rangle^2}{\langle Ba \rangle^2} \frac{\tau_n(a^-, B^-, -C^+, d^+, \dots, n^+)}{\ell^2 B^2 C^2} \quad (3.5)$$

A factor of  $\langle bc \rangle^{-1}$  arises from the region of integration where  $\ell^2 = 0$ . Specifically, since  $B^2 = \ell^2 + 2\ell \cdot b + b^2 = \ell^2 + 2\ell \cdot b$ , around  $\ell^2 = 0$ ,

$$C^{\text{n-s f}} \sim \int_0 |\ell|^{d-1} d\ell \frac{\tau|_{\ell^2=0}}{\ell^2(\ell^2 + 2\ell \cdot b)(\ell^2 - 2\ell \cdot c)} \sim \int_0 |\ell|^{d-1} \frac{d\ell}{\ell^2(\ell \cdot b)(\ell \cdot c)} \sim \frac{1}{\langle bc \rangle} \quad (3.6)$$

We can expand  $\tau_n$  into sub-currents which are either MHV currents or currents with a single minus as show in fig. 3. The first structure in fig. 3 we label  $\tau_n^{\text{tri}}$ . This contains an

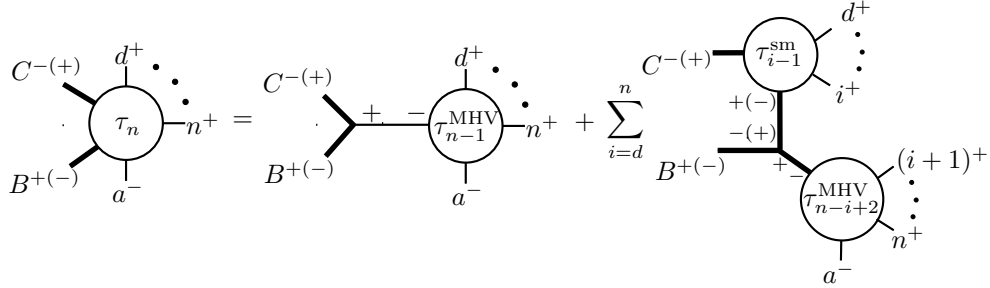


FIG. 3: An organisation of the diagrams within  $\tau_n$ . We have used the vanishing of the specific single minus current (3.7).

explicit pole and generates a further pole upon integration to give rise to the double pole contributions. The other structures only generate single poles and we label them  $\tau_n^b$ . The diagrammatic expansion gives both of these contributions in terms of off-shell MHV tree currents:  $\tau_n^{\text{MHV}}$ . We can use the general results, specialised to  $\lambda_q = \lambda_a$ , for the currents with

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external legs from  $n_3$  three-point MHV vertices,  $\bar{n}_3$  three point  $\overline{\text{MHV}}$  vertices and  $n_4$  four-point MHV vertices. Then  $n_- = n_3 + n_4$  and  $n_+ = \bar{n}_3 + n_4$ . For our situation with  $n_- = 1$  there is thus either a single three or four point MHV vertex.

one off-shell momentum,  $P$ , given by [13, 20],

$$\tau^{\text{sm}}(a^-, P^+, \dots, n^+) = 0 \quad (3.7)$$

$$\tau^{\text{sm}}(P^-, 1^+, \dots, n^+) = -\frac{P^2 \langle Pa \rangle^2}{\langle a1 \rangle \langle 12 \rangle \dots \langle n-1, n \rangle \langle na \rangle} \quad (3.8)$$

$$\tau^{\text{MHV}}(a^-, P^-, f^+ \dots n^+) = \frac{i \langle Pa \rangle^2 P^2}{\langle af \rangle \langle fg \rangle \dots \langle (n-1)n \rangle \langle na \rangle} \left( \frac{[qn]}{[aq][an]} - \sum_{l=f}^{n-1} \frac{\langle a|P_l l|a \rangle}{P_l^2 P_{l-1}^2} \right) \quad (3.9)$$

where  $P_l \equiv k_{l+1} + \dots + k_n + k_a$ . Although the last expression contains an explicit  $P^2$  factor in the numerator, the term in the sum with  $l = f$  contains a  $1/P^2$  since  $P_{f-1} = -P$  and so survives in the  $P^2 \rightarrow 0$  limit. We will need a simple generalisation of (3.8),

$$\tau^{\text{sm}}(P^{-h}, f^{+h} \dots n^+) = \left( \frac{\langle fa \rangle}{\langle Pa \rangle} \right)^{2-2h} \tau^{\text{sm}}(P^-, f^+ \dots n^+) \quad (3.10)$$

This result follows diagram by diagram in  $\tau^{\text{sm}}$  as only three-point  $\overline{\text{MHV}}$  vertices are present and every one that the non-gluonic particle encounters introduces a factor of

$$\left( \frac{[\kappa p_{\text{in}}]}{[\kappa p_{\text{out}}]} \right)^{2-2h} = \left( \frac{\langle p_{\text{out}} a \rangle}{\langle p_{\text{in}} a \rangle} \right)^{2-2h} \left( \frac{[\kappa | p_{\text{in}} | a]}{[\kappa | p_{\text{out}} | a]} \right)^{2-2h} = \left( \frac{\langle p_{\text{out}} a \rangle}{\langle p_{\text{in}} a \rangle} \right)^{2-2h}, \quad (3.11)$$

the product of which gives the factor in (3.10) for each diagram.

For  $\tau_n^{\text{tri}}$  we have

$$\begin{aligned} \tau_n^{\text{tri}} &= V_3(P^+, B^+, -C^-) \frac{1}{P_{BC}^2} \tau_{n-1}^{\text{MHV}}(a^-, -P^-, d^+, \dots, n^+) \\ &= \frac{\langle Ca \rangle^2}{\langle Ba \rangle^2} \frac{\langle a|BC|a \rangle}{s_{bc}} \frac{\tau_{n-1}^{\text{MHV}}(a^-, (b+c)^-, d^+, \dots, n^+)}{\langle P_{b+c} a \rangle^2} \end{aligned} \quad (3.12)$$

This has a very simple dependence on the off-shell momenta  $B$  and  $C$  and contains an explicit  $1/s_{bc}$  factor which, together with the pole arising from the integration, produces the double pole. This term also contributes to the subleading pole.

The remaining contributions to  $\tau_n$  arise from the second class of diagram in fig. 3. The integrand does not have an explicit pole as  $\langle bc \rangle \rightarrow 0$  and only generates a single pole after integration. Since this arises at  $C^2 = B^2 = 0$ , we can take  $C^2 = 0$  so that the  $\tau^{\text{sm}}$  structures in fig. 3 have only one massive leg. In this limit we can use the formulae of equations (3.8) and (3.9) for currents with a single massive leg to obtain (to leading order in  $\langle bc \rangle$ )

$$\tau_n^b = \frac{\langle Ca \rangle^2}{\langle Ba \rangle^2} \left[ \frac{\langle ca \rangle \langle b|l|a \rangle}{\langle cd \rangle \langle de \rangle \dots \langle na \rangle \langle ab \rangle} - \sum_{i=d}^{n-1} \frac{\langle ca \rangle \langle a|BK_i|a \rangle}{\langle cd \rangle \langle de \rangle \dots \langle ia \rangle} \frac{\tau^{\text{MHV}}(a^-, -K_i^-, (i+1)^+, \dots, n^+)}{\langle K_i a \rangle^2 K_i^2} \right] \quad (3.13)$$

where the  $K_i = k_{i+1} + \dots + k_n + k_a$  are fixed by momentum conservation within the  $\tau^{\text{MHV}}$



structures and we have made use of:

$$\frac{\langle Ca \rangle}{\langle Cd \rangle} = \frac{\langle ca \rangle}{\langle cd \rangle} + \mathcal{O}(\langle bc \rangle) \quad (3.14)$$

in the relevant integration region. In this form we see that all of the contributions to  $C^{\text{n-sf}}$  involve the same basic integral:

$$\int d^4\ell \frac{[b|\ell|a][c|\ell|a][X|B|a]}{\ell^2 B^2 C^2} = \frac{i}{96\pi^2} \frac{\langle a|bc|a \rangle}{\langle bc \rangle} [X|2b+c|a] \quad (3.15)$$

leading to

$$\begin{aligned} & C^{\text{n-sf}}(a^-, b^+, c^+, d^+, e^+, \dots, n^+) \\ &= \frac{i}{96\pi^2} \frac{[bc]}{\langle bc \rangle} \left[ \frac{\langle a|\beta(b+c)|a \rangle}{s_{bc}} \frac{\tau^{\text{MHV}}(a^-, (b+c)^-, d^+ \dots n)}{\langle P_{b+c}a \rangle^2} + \frac{\langle ca \rangle [b|\beta|a]}{\langle cd \rangle \langle de \rangle \dots \langle na \rangle [ab]} \right. \\ & \quad \left. - \sum_{i=d}^{n-1} \frac{\langle ca \rangle \langle a|\beta K_i|a \rangle}{\langle cd \rangle \langle de \rangle \dots \langle ia \rangle} \frac{\tau^{\text{MHV}}(a^-, -K_i^-, (i+1)^+, \dots, n^+)}{\langle K_i a \rangle^2 K_i^2} \right] \end{aligned} \quad (3.16)$$

where  $\beta = 2b + c$ . Setting  $\gamma = -b$  so that  $\beta + \gamma = b + c$ , we have

$$\frac{\langle \gamma a \rangle}{\langle \gamma d \rangle} = \frac{\langle ba \rangle}{\langle bd \rangle} \frac{\langle cd \rangle}{\langle cd \rangle} = \frac{\langle ca \rangle}{\langle cd \rangle} + \mathcal{O}(\langle bc \rangle), \quad (3.17)$$

so to leading order in  $\langle bc \rangle$  we have

$$\begin{aligned} & C^{\text{n-sf}}(a^-, b^+, c^+, d^+, e^+, \dots, n^+) \\ &= \frac{i}{96\pi^2} \frac{[bc]}{\langle bc \rangle} \left[ \frac{\langle a|\beta(b+c)|a \rangle}{s_{bc}} \frac{\tau^{\text{MHV}}(a^-, (b+c)^-, d^+ \dots n)}{\langle P_{b+c}a \rangle^2} + \frac{\langle \gamma a \rangle [b|\beta|a]}{\langle \gamma d \rangle \langle de \rangle \dots \langle na \rangle [ab]} \right. \\ & \quad \left. + \sum_{i=d}^{n-1} \frac{\langle a|\beta \kappa_i|a \rangle \langle \gamma a \rangle^2}{K_i^2 \kappa_i^2} \frac{\tau^{\text{sm}}(-\kappa_i^-, \gamma^+, d^+ \dots i^+)}{\langle \kappa_i a \rangle^2} \frac{\tau^{\text{MHV}}(a^-, -K_i^-, (i+1)^+ \dots n)}{\langle K_i a \rangle^2} \right] \end{aligned} \quad (3.18)$$

Again the internal momenta,  $\kappa_i$ , are specified by momentum conservation within the  $\tau^{\text{sm}}$  factors. The quantity in the square brackets is now essentially  $\tau_n(a^-, \beta^-, \gamma^+, \dots, n^+)$ . For clarity we can absorb the second term into the summation by adopting an appropriate definition for  $\tau^{\text{MHV}}(a^-, K_n^-)$ .

Using the explicit form for  $\tau^{\text{MHV}}$  in the  $\tau^{\text{tri}}$  contribution we have, to leading order,

$$\begin{aligned}
& C^{\text{n-sf}}(a^-, b^+, c^+, d^+, e^+, \dots, n^+) \\
&= \frac{i}{96\pi^2} \frac{[bc]}{\langle bc \rangle} \left[ \frac{\langle a|\beta(b+c)|a\rangle}{\langle ad\rangle\langle de\rangle \dots \langle (n-1)n\rangle\langle na\rangle} \left( \frac{[qn]}{[aq][an]} + \frac{\langle a|(b+c)d|a\rangle}{t_{bcd}s_{bc}} - \sum_{l=e}^{n-1} \frac{\langle a|K_{a,l+1..n}l|a\rangle}{s_{a,l+1..n}s_{a,l..n}} \right) \right. \\
&\quad \left. + \sum_{i=d}^n \frac{\langle a|\beta\kappa_i|a\rangle\langle\gamma a\rangle^2}{K_i^2\kappa_i^2} \frac{\tau^{\text{sm}}(-\kappa_i^-, \gamma^+, d^+ \dots i^+)}{\langle\kappa_i a\rangle^2} \frac{\tau^{\text{MHV}}(a^-, -K_i^-, (i+1)^+ \dots n)}{\langle K_i a\rangle^2} \right]
\end{aligned} \tag{3.19}$$

As discussed previously, the contribution to the rational term is

$$\text{Res} \left( \frac{1}{z} C^{\text{n-sf}}(\hat{a}^-, \hat{b}^+, c^+, d^+, e^+, \dots, n^+) \right) \Big|_{\langle \hat{bc} \rangle = 0} \tag{3.20}$$

To extract the residue of the double pole term we use:

$$\text{Res} \left( \frac{[bc]}{\langle \hat{bc} \rangle} \frac{1}{z t_{bcd} s_{bcd}} \right) \Big|_{\langle \hat{bc} \rangle = 0} = \frac{1}{\langle bc \rangle \langle a|(b+c)d|c \rangle} \left( \frac{\langle ac \rangle}{\langle bc \rangle} - \frac{[b|c+d|a]\langle ac \rangle}{\langle a|(b+c)d|c \rangle} \right) \tag{3.21}$$

The first term here is precisely the double pole contribution of [14]

$$A_{n-1}^{(0)}(d^+, \dots, n^+, \hat{a}^-, \hat{K}_{bc}^-) \frac{i}{(K_{bc}^2)^2} V^{1\text{-loop}}(\hat{b}^+, c^+, -\hat{K}_{bc}^+) = \frac{i}{96\pi^2} \frac{\langle ac \rangle}{\langle bc \rangle^2} \frac{\langle a|\beta(b+c)|a\rangle}{\langle cd\rangle\langle de\rangle \dots \langle (n-1)n\rangle\langle na\rangle} \tag{3.22}$$

The second term in (3.21) contains only a single factor of  $\langle bc \rangle$  in the denominator and its coefficient is unaffected by the shift. We combine this with the single pole pieces of (3.19) to write the full sub-leading or *pole under the pole* (PUP) contribution as:

$$\begin{aligned}
C^{\text{PUP}} &= \frac{i}{96\pi^2} \frac{[bc]}{\langle bc \rangle} \\
&\times \left[ \frac{\langle a|\beta(b+c)|a\rangle}{\langle ad\rangle\langle de\rangle \dots \langle (n-1)n\rangle\langle na\rangle} \left( \frac{[qn]}{[aq][an]} - \frac{\langle a|(b+c)d|a\rangle[b|c+d|a]\langle ac \rangle}{[bc]\langle a|(b+c)d|c \rangle^2} - \sum_{l=e}^{n-1} \frac{\langle a|K_{a,l+1..n}l|a\rangle}{s_{a,l+1..n}s_{a,l..n}} \right) \right. \\
&\quad \left. + \sum_{i=d}^n \frac{\langle a|\beta\kappa_i|a\rangle\langle\gamma a\rangle^2}{K_i^2\kappa_i^2} \frac{\tau^{\text{sm}}(-\kappa_i^-, \gamma^+, d^+ \dots i^+)}{\langle\kappa_i a\rangle^2} \frac{\tau^{\text{MHV}}(a^-, -K_i^-, (i+1)^+ \dots n)}{\langle K_i a\rangle^2} \right]_{\tilde{\dagger}}
\end{aligned} \tag{3.23}$$

where  $\tilde{\dagger}$  denotes that the quantity in square brackets is to be shifted and evaluated at  $z = -\langle bc \rangle / \langle ac \rangle$ . The sums in this expression are most of the terms in the expansion of an on-shell MHV amplitude. We can use the simple interchange properties of the Parke-Taylor amplitudes and  $\tau^{\text{sm}}$  to gather many of these terms into the finite on-shell MHV amplitude

$\tau^{\text{MHV}}(a^-, \beta^-, d^+, \gamma^+, e^+ \dots n^+)$ . From a diagrammatic perspective we have,

$$\begin{aligned} \tau^{\text{MHV}}(a^-, \beta^-, d^+, \gamma^+, e^+ \dots n^+) &= \frac{[d|K_1|a]\langle\beta a\rangle^2}{\langle da\rangle} \frac{\tau^{\text{MHV}}(a^-, -K_1^-, \gamma^+, e^+ \dots n)}{\langle K_1 a\rangle^2 K_1^2} \\ &+ \langle a|\beta\kappa_2|a\rangle\langle\beta a\rangle^2 \frac{\tau^{\text{sm}}(-\kappa_2^-, d^+, \gamma^+)}{\langle\kappa_2 a\rangle^2 \kappa_2^2} \frac{\tau^{\text{MHV}}(a^-, -K_2^-, e^+ \dots n)}{\langle K_2 a\rangle^2 K_2^2} \\ &+ \sum_{i=e}^n \langle a|\beta\kappa_i|a\rangle\langle\beta a\rangle^2 \frac{\tau^{\text{sm}}(-\kappa_i^-, d^+, \gamma^+, e, \dots i^+)}{\langle\kappa_i a\rangle^2 \kappa_i^2} \frac{\tau^{\text{MHV}}(a^-, -K_i^-, (i+1)^+ \dots n)}{\langle K_i a\rangle^2 K_i^2}. \end{aligned} \quad (3.24)$$

Interchanging the  $\gamma$  and  $d$  legs in the  $\tau^{\text{sm}}$  amplitudes in the final sum introduces a simple factor of the form  $-(\langle ad\rangle/\langle a\gamma\rangle)(\langle\gamma e\rangle/\langle de\rangle)$ . We can then replace all but one term of the final sum in (3.23) with  $\tau^{\text{MHV}}(a^-, \beta^-, d^+, \gamma^+, e^+ \dots n^+)$  and the first two terms in the expansion (3.24):

$$\begin{aligned} \mathcal{C}^{\text{PUP}} &= \frac{i}{96\pi^2} \frac{[bc]}{\langle bc\rangle} \\ &\times \left[ \frac{\langle a|\beta(b+c)|a\rangle}{\langle ad\rangle\langle de\rangle \dots \langle (n-1)n\rangle\langle na\rangle} \left( \frac{[qn]}{[aq][an]} - \frac{\langle a|(b+c)d|a\rangle[b|c+d|a]\langle ac\rangle}{[bc]\langle a|(b+c)d|c\rangle^2} - \sum_{l=e}^{n-1} \frac{\langle a|K_{a,l+1..n}l|a\rangle}{s_{a,l+1..n}s_{a,l..n}} \right) \right. \\ &+ \frac{\langle a|\beta\kappa_2|a\rangle\langle\gamma a\rangle^2}{K_2^2\kappa_2^2} \frac{\tau^{\text{sm}}(-\kappa_2^-, \gamma^+, d^+)}{\langle\kappa_2 a\rangle^2} \frac{\tau^{\text{MHV}}(a^-, -K_2^-, e^+ \dots n^+)}{\langle K_2 a\rangle^2} \\ &- \frac{\langle ad\rangle\langle\gamma e\rangle}{\langle a\gamma\rangle\langle de\rangle} \left( \frac{[d|K_1|a]\langle\gamma a\rangle^2}{\langle da\rangle} \frac{\tau^{\text{MHV}}(a^-, -K_1^-, \gamma^+, e^+ \dots n)}{K_1^2\langle K_1 a\rangle^2} \right. \\ &\quad + \frac{\langle a|\beta\kappa_2|a\rangle\langle\gamma a\rangle^2}{K_2^2\kappa_2^2} \frac{\tau^{\text{sm}}(-\kappa_2^-, d^+, \gamma^+)}{\langle\kappa_2 a\rangle^2} \frac{\tau^{\text{MHV}}(a^-, -K_2^-, e^+ \dots n^+)}{\langle K_2 a\rangle^2} \\ &\quad \left. \left. - \frac{\langle\gamma a\rangle^2}{\langle\beta a\rangle^2} \tau^{\text{MHV}}(a^-, \beta^-, d^+, \gamma^+, e^+ \dots n^+) \right) \right]_{\tilde{\text{f}}} \end{aligned} \quad (3.25)$$

The shift puts the final  $\tau^{\text{MHV}}$  on-shell and we can use the Parke-Taylor form for this term. Using (3.9) for the off-shell  $\tau^{\text{MHV}}$  factors we see that all of these contain a common term of

$$\frac{[qn]}{[aq][an]} - \sum_{l=e}^{n-1} \frac{\langle a|K_{a,l+1..n}l|a\rangle}{s_{a,l+1..n}s_{a,l..n}} \quad (3.26)$$

The overall coefficient of this term vanishes before we apply the shift. The sum in (3.9) for  $\tau^{\text{MHV}}(a^-, K_1^-, \gamma^+, e^+ \dots n)$  also contains an  $l = \gamma$  term. On the pole this cancels with the contribution from the second term in (3.25) leaving just the  $\tau^{\text{MHV}}(a^-, \beta^-, d^+, \gamma^+, e^+ \dots n^+)$  term,

$$\mathcal{C}^{\text{PUP}} = \frac{i}{96\pi^2} \frac{[bc]}{\langle bc\rangle} \times \left[ \frac{\langle ad\rangle\langle\gamma e\rangle}{\langle a\gamma\rangle\langle de\rangle} \frac{\langle\gamma a\rangle^2}{\langle\beta a\rangle^2} \tau^{\text{MHV}}(a^-, \beta^-, d^+, \gamma^+, e^+ \dots n^+) \right]_{\tilde{\text{f}}} \quad (3.27)$$

Performing the shift, under which  $\lambda_\beta, \lambda_\gamma \rightarrow \lambda_c$  at the pole, we obtain

$$\mathcal{C}^{\text{PUP}} = \frac{i}{96\pi^2} \frac{[bc]}{\langle bc \rangle} \times \frac{\langle ad \rangle \langle ce \rangle}{\langle ac \rangle \langle de \rangle} \tau^{\text{MHV}}(a^-, c^-, d^+, c^+, e^+ \dots n^+) \quad (3.28)$$

which exactly reproduces the soft factor of ref.[14] given in eq. (2.4).

#### IV. FIVE GRAVITON AMPLITUDE

A one-loop graviton scattering amplitude can receive contributions from a range of particle types circulating in the loop. We denote the contribution from a particle of spin- $s$  to the graviton scattering amplitude by  $M_n^{[s]}$  (with  $M_n^{[0]}$  representing a real scalar). In a supergravity theory there can be contributions from minimally coupled matter multiplets. The contributions from the various supergravity multiplets are [21]<sup>5</sup>

$$\begin{aligned} M_n^{\mathcal{N}=8} &= M_n^{[2]} + 8M_n^{[3/2]} + 28M_n^{[1]} + 56M_n^{[1/2]} + 70M_n^{[0]} \\ M_n^{\mathcal{N}=6, \text{matter}} &= M_n^{[3/2]} + 6M_n^{[1]} + 15M_n^{[1/2]} + 20M_n^{[0]} \\ M_n^{\mathcal{N}=4, \text{matter}} &= M_n^{[1]} + 4M_n^{[1/2]} + 6M_n^{[0]} \\ M_n^{\mathcal{N}=1, \text{matter}} &= M_n^{[1/2]} + 2M_n^{[0]} \end{aligned} \quad (4.1)$$

These relations can be inverted to obtain a supersymmetric decomposition of the pure graviton scattering amplitude,

$$M_n^{[2]} = M_n^{\mathcal{N}=8} - 8M_n^{\mathcal{N}=6, \text{matter}} + 20M_n^{\mathcal{N}=4, \text{matter}} - 16M_n^{\mathcal{N}=1, \text{matter}} + 2M_n^{[0]} \quad (4.2)$$

Compared to Yang-Mills theory, the one-loop amplitudes for graviton scattering are relatively poorly understood. Previously, only for  $n = 4$  have all the components of (4.2) been computed [21–23]. For  $n = 5$ , the purely rational amplitudes  $M_5(++++)$  and  $M_5(-++++)$  only have non-vanishing scalar components which have been computed in refs.[24] and [25] respectively. For the MHV amplitude  $M_5(--+++)$  only the supersymmetric components have been computed previously: the  $\mathcal{N} = 8$  component in ref [24], the  $\mathcal{N} = 6$  in [26] and the  $\mathcal{N} = 4, 1$  components in [27, 28]. In this section we use complex factorisation to obtain the last remaining component,  $M_5^{[0]}$ , of the five graviton scattering amplitude.

Our starting point is (1.8) and its the apparently trivial rewriting:

$$R_n = \sum (\text{Feynman Diagrams}) - \sum_{i \in \mathcal{C}} c_i I_{4:\text{trunc}}^i - \sum_{k \in \mathcal{E}} e_k I_2^k, \quad (4.3)$$

where we have absorbed the triangle integral contributions into the *truncated* box contributions [27, 29] and identified the amplitude with a sum of Feynman diagrams. The coefficients of the box and bubble integral functions in (4.3) are obtained using four dimensional unitarity methods and we then perform a BCFW [4] recursion on (4.3) to obtain  $R_5$ . Applying a BCFW shift

$$\bar{\lambda}_1 \rightarrow \bar{\lambda}_1 = \bar{\lambda}_1 - z\bar{\lambda}_3, \quad \lambda_3 \rightarrow \lambda_3 = \lambda_3 + z\lambda_1, \quad (4.4)$$

---

<sup>5</sup> We use the normalisation for the full physical amplitudes  $\mathcal{M}^{\text{tree}} = i(\kappa/2)^{n-2} M^{\text{tree}}, \mathcal{M}^{1\text{-loop}} = i(2\pi)^{-2}(\kappa/2)^n M^{1\text{-loop}}$ .

we obtain

$$R_5 = \sum_{z \neq 0 \text{ poles}} \text{Res} \left( \frac{1}{z} \left( M(z) - \sum_{i \in \mathcal{C}} \tilde{c}_i I_{4:\text{trunc}}^i - \sum_{k \in \mathcal{E}} e_k I_2^k \right) \right) \quad (4.5)$$

subject to the usual caveat about the behaviour of  $M(z)$  at large  $z$ . We denote the contributions arising from the diagrams, boxes and bubbles as  $R^{\text{diag}}$ ,  $R^{\text{box}}$  and  $R^{\text{bub}}$  respectively, so that,

$$R_5 = R_5^{\text{diag}} + R_5^{\text{box}} + R_5^{\text{bub}}. \quad (4.6)$$

The rational *descendants* of the box and bubble contributions are obtained by expanding them around whatever physical or spurious singularities they contain, while the poles in the Feynman diagrams correspond to the standard factorisations of the amplitude and the non-standard factorisations discussed previously.

### A. Factorisations

The diagrammatic contribution has two parts: the standard and non-standard factorisations. We therefore set

$$R_5^{\text{diag}} = R_5^{\text{st}} + R_5^{\text{ns}} \quad (4.7)$$

As there are non-vanishing one-loop scalar single-minus amplitudes, the five-point amplitude has standard factorisations (2.1) of the form

$$\begin{aligned} & M^{\text{tree}}(a^-, b^-, -P^+) \frac{1}{P^2} M^{1\text{-loop}}(P^-, c^+, d^+, e^+), \\ & M^{\text{tree}}(a^-, c^+, -P^+) \frac{1}{P^2} M^{1\text{-loop}}(P^-, b^-, d^+, e^+), \\ & M^{\text{tree}}(a^-, c^+, -P^-) \frac{1}{P^2} M^{1\text{-loop}}(b^-, P^+, d^+, e^+), \\ & M^{\text{tree}}(c^+, d^+, -P^-) \frac{1}{P^2} M^{1\text{-loop}}(a^-, b^-, P^+, e^+). \end{aligned} \quad (4.8)$$

These contain simple poles so the rational contributions come from the rational parts of the four-point amplitudes. The shift (4.4) excites six different poles, leading to

$$\begin{aligned} R_5^{\text{st}} = & \frac{1}{s_{12}} \left( M^{\text{tree}}(\hat{1}^-, 2^-, -P^+) \times R_4(P^-, \hat{3}^+, 4^+, 5^+) \right) \Big|_{s_{12}=0} \\ & + \frac{1}{s_{14}} \left( M^{\text{tree}}(\hat{1}^-, -P^-, 4^+) \times R_4(2^-, P^+, \hat{3}^+, 5^+) \right) \Big|_{s_{14}=0} \\ & + \frac{1}{s_{15}} \left( M^{\text{tree}}(\hat{1}^-, -P^-, 5^+) \times R_4(2^-, P^+, \hat{3}^+, 4^+) \right) \Big|_{s_{15}=0} \\ & + \frac{1}{s_{23}} \left( M^{\text{tree}}(2^-, \hat{3}^+, -P^+) \times R_4(\hat{1}^-, P^-, 4^+, 5^+) \right) \Big|_{s_{23}=0} \\ & + \frac{1}{s_{34}} \left( M^{\text{tree}}(-P^-, \hat{3}^+, 4^+) \times R_4(\hat{1}^-, 2^-, P^+, 5^+) \right) \Big|_{s_{34}=0} \\ & + \frac{1}{s_{35}} \left( M^{\text{tree}}(-P^-, \hat{3}^+, 5^+) \times R_4(\hat{1}^-, 2^-, P^+, 4^+) \right) \Big|_{s_{35}=0} \end{aligned} \quad (4.9)$$

where the rational parts of the four-point amplitudes are [21]:

$$\begin{aligned}
R_4(a^-, b^+, c^+, d^+) &= \frac{1}{360} \left( \frac{s_{ab}s_{ad}}{s_{ac}} \right)^2 \left( \frac{[bd]^2}{[ab]\langle bc\rangle\langle cd\rangle[da]} \right)^2 (s_{ab}^2 + s_{ab}s_{ad} + s_{ad}^2) \\
R_4(a^-, b^-, c^+, d^+) &= \frac{1}{360s_{ab}^6} \left( \frac{s_{ab}s_{ad}\langle ab\rangle^3}{\langle bc\rangle\langle cd\rangle\langle da\rangle} \right)^2 (2s_{ad}^4 + 23s_{ac}s_{ad}^3 + 222s_{ac}^2s_{ad}^2 + 23s_{ac}^3s_{ad} + 2s_{ac}^4)
\end{aligned} \tag{4.10}$$

There are also non-standard factorisations. As discussed in section 3, we expect a complex pole when adjacent massless legs on a loop become collinear as in fig. 4.

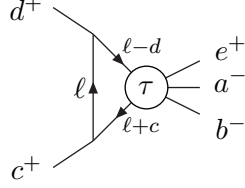


FIG. 4: The non-standard factorisation diagram at five-point

At five-point the tree current,  $\tau$ , is  $\overline{\text{MHV}}$  which does not diverge in the  $\langle cd \rangle \rightarrow 0$  limit. We therefore only find  $\langle cd \rangle^{-1}$  poles in this case. The region of interest has all three of the illustrated propagators close to null, so to leading order we can replace  $\tau$  by an on-shell tree amplitude

$$\int d^d\ell \left( \frac{[c|\ell|q]\langle d|\ell|q\rangle}{\langle cq\rangle\langle dq\rangle} \right)^2 \frac{\tau(e^+, A, B, a^-, b^-)}{\ell^2 A^2 B^2} \rightarrow \int d^d\ell \left( \frac{[c|\ell|q]\langle d|\ell|q\rangle}{\langle cq\rangle\langle dq\rangle} \right)^2 \frac{M^{\text{tree}}(e^+, A, B, a^-, b^-)}{\ell^2 A^2 B^2}, \tag{4.11}$$

where  $A = \ell + c$ ,  $B = -\ell + d$ . We use the Kawai-Lewellen-Tye relations [30] to express the gravity tree amplitude in terms of Yang-Mills amplitudes,

$$\begin{aligned}
M^{\text{tree}}(e^+, A, B, a^-, b^-) &= s_{AB}s_{ab}A^{\text{tree}}(e^+, A, B, a^-, b^-)A^{\text{tree}}(e^+, B, A, b^-, a^-) \\
&\quad + s_{Aa}s_{Bb}A^{\text{tree}}(e^+, A, a^-, B, b^-)A^{\text{tree}}(e^+, a^-, A, b^-, B) \\
&= \frac{[eB]^4[eA]^4\langle AB\rangle\langle ab\rangle}{[eA][Ba][be][eB][BA][Ab][ba][ae]} + \frac{[eB]^4[eA]^4\langle Aa\rangle\langle Bb\rangle}{[eA][aB][be][ea][aA][Ab][bB][Be]}
\end{aligned} \tag{4.12}$$

In the region of interest  $\langle AB \rangle \sim 0$ , so the first term is negligible, leading to the contribution to  $R_5^{\text{ns}}$ :

$$R_{5:cd}^{\text{ns}} = \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{1}{[ae][be]\langle cq\rangle^2\langle dq\rangle^2} \frac{[c|\ell|q]^2[d|\ell|q]^2[eB]^3[eA]^3\langle Aa\rangle\langle Bb\rangle}{[aB][aA][Ab][bB]} + \mathcal{O}(\langle cd \rangle) \tag{4.13}$$

For the five-point amplitude we only need the leading term on the pole which suggests that the details of the off-shell continuation of  $\ell$ ,  $A$  and  $B$  are not important. However the integration region may contain points where one of  $[aA]$ ,  $[bA]$ ,  $[aB]$  or  $[bB]$  diverge. As the only genuine IR divergences in a diagrammatic formulation arise from propagators, we

rewrite all such factors in the denominator as propagators using:

$$\frac{1}{[xA]} = \frac{\langle Ax \rangle}{[xA]\langle Ax \rangle} = \frac{\langle Ax \rangle}{(A+x)^2} + \mathcal{O}(\langle cd \rangle), \quad (4.14)$$

where the final step involves introducing a sub-leading piece ( $A^2$  in this case). With the labelling in the figure, the possible propagators involving  $A$ ,  $B$ ,  $a$  and  $b$  are;

$$\frac{1}{(A+a)^2}, \quad \frac{1}{(A+b)^2}, \quad \frac{1}{(B-a)^2}, \quad \frac{1}{(B-b)^2}, \quad (4.15)$$

which fixes the ambiguity with regard to using  $(A \pm a)^{-2}$ . Making use of  $\langle Ax \rangle \langle By \rangle = \langle Ay \rangle \langle Bx \rangle + \mathcal{O}(\langle cd \rangle)$  and setting  $\lambda_q = \lambda_a$  we have,

$$R_{5:cd}^{\text{ns}} = \int \frac{d^d \ell}{\ell^2 A^2 B^2} \frac{1}{[ae][be]\langle ca \rangle^2 \langle da \rangle^2} \frac{[c|A|b]^2 [d|B|b] [d|B|a] [e|B|a]^3 [e|A|a]^3}{(B-a)^2 (B-b)^2 (A+a)^2 (A+b)^2} + \mathcal{O}(\langle cd \rangle) \quad (4.16)$$

We now apply some leading order reductions:

$$[c|A|b][ba]\langle a|A|e \rangle = -2(b \cdot A)[c|Aa|e] + 2(a \cdot A)[c|Ab|e] + \mathcal{O}(\langle cd \rangle), \quad (4.17)$$

$$[d|B|b][ba]\langle a|B|e \rangle = -2(b \cdot B)[d|Ba|e] + 2(a \cdot B)[d|Bb|e] + \mathcal{O}(\langle cd \rangle), \quad (4.18)$$

leading to,

$$R_{5:cd}^{\text{ns}} = I_{a,b}^{[0]} + I_{b,a}^{[0]} - I_{a,a}^{[0]} - I_{b,b}^{[0]} + \mathcal{O}(\langle cd \rangle), \quad (4.19)$$

where

$$I_{x,y}^{[0]} = \int \frac{d^d \ell}{\ell^2 A^2 B^2} \frac{1}{[ab]^2 [ae][be]\langle ca \rangle^2 \langle da \rangle^2} \frac{[c|A|b][d|B|a][e|B|a]^2 [e|A|a]^2 [c|Ay|e][d|Bx|e]}{(B-x)^2 (A+y)^2}. \quad (4.20)$$

The terms with  $x = y$  reduce directly to box integrals using

$$\frac{1}{(B-x)^2 (A+x)^2} = \frac{1}{2P \cdot x} \left( \frac{1}{(B-x)^2} - \frac{1}{(A+x)^2} \right) + \mathcal{O}(\langle cd \rangle). \quad (4.21)$$

Giving  $I_{x,x}^{[0]} = I_{x,x:A}^{[0]} + I_{x,x:B}^{[0]} + \mathcal{O}(\langle cd \rangle)$ , with

$$I_{x,x:A}^{[0]} = \int \frac{d^d \ell}{\ell^2 A^2 B^2} \frac{-1}{[ab]^2 [ae][be]\langle ca \rangle^2 \langle da \rangle^2 2(P \cdot x)} \frac{[c|A|b][d|B|a][e|B|a]^2 [e|A|a]^2 [c|Ax|e][d|Bx|e]}{(A+x)^2} \quad (4.22)$$

$$I_{x,x:B}^{[0]} = \int \frac{d^d \ell}{\ell^2 A^2 B^2} \frac{1}{[ab]^2 [ae][be]\langle ca \rangle^2 \langle da \rangle^2 2(P \cdot x)} \frac{[c|A|b][d|B|a][e|B|a]^2 [e|A|a]^2 [c|Ax|e][d|Bx|e]}{(B-x)^2} \quad (4.23)$$

For  $x \neq y$  successive reductions give

$$I_{x,\bar{x}}^{[0]} = \sum_{i=1}^5 (I_{x,\bar{x}:A_i}^{[0]} + I_{x,\bar{x}:B_i}^{[0]}) + I_{x,\bar{x}:P}^{[0]} + \mathcal{O}(\langle cd \rangle), \quad (4.24)$$

where the terms within the sum are the box integrals given explicitly in appendix A and the

final term is a cubic pentagon which does not contribute to the rational term:

$$I_{x,\bar{x};P}^{[0]} = \frac{[ex]^2[e\bar{x}]^2[c\bar{x}][xd][\bar{x}|P|x][x|P|\bar{x}]}{[ab]^6\langle ca\rangle^2\langle da\rangle^2[\bar{x}x]} \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{[c|A|a][d|B|a]\langle a|P\bar{x}ABxP|a\rangle}{(B-x)^2(A+\bar{x})^2}. \quad (4.25)$$

In general, although the non-standard factorisations give both rational and transcendental contributions, we are only interested in the former. We take  $I_{x,\bar{x};B_5}^{[0]}$  as an example:

$$I_{x,\bar{x};B_5}^{[0]} = \frac{[ex]^2[e\bar{x}]^2[c\bar{x}][xd][\bar{x}|P|x][x|P|\bar{x}][x|P|a]}{[ab]^6\langle ca\rangle^2\langle da\rangle^2[\bar{x}x]} \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{[c|A|a][d|B|a]\langle x|BP|a\rangle}{(B-x)^2}. \quad (4.26)$$

We are interested in the rational term generated by the box integral of fig. 5 with the numerator given in (4.26).

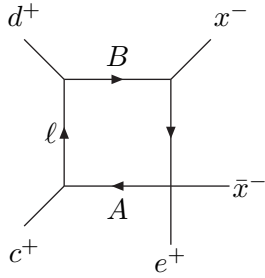


FIG. 5: The box integral  $I_{x,\bar{x};B_5}^{[0]}$

As discussed previously, we expect this box integral to have a  $\langle cd\rangle^{-1}$  singularity. Dropping the pre-factor, the  $\ell$  dependent part of the integral is

$$I_{x,\bar{x};B_5}^{\text{bare}} = \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{[c|A|a][d|B|a]\langle x|BP|a\rangle}{(B-x)^2}, \quad (4.27)$$

Appealing to (1.8), any divergences in this integral will be contained in the cut-constructible pieces so the rational terms of interest here are finite. Splitting the integration into a four dimensional part and an  $\epsilon$  dimensional part [31] we have,

$$I_{x,\bar{x};B_5}^{\text{bare}} = -\epsilon\pi^{-\epsilon}\Gamma(1-\epsilon) \times \int_0^\infty d\mu^2 \int \frac{(\mu^2)^{-1-\epsilon} d^4\ell}{(\ell^2 - \mu^2)(A^2 - \mu^2)(B^2 - \mu^2)} \frac{[c|A|a][d|B|a]\langle x|BP|a\rangle}{(B-x)^2 - \mu^2}, \quad (4.28)$$

where all the momenta are now four dimensional. Parametrising the four dimensional loop momentum by

$$\ell = \alpha k_c + \beta k_d + \gamma \bar{\lambda}_c \lambda_d + \bar{\gamma} \bar{\lambda}_d \lambda_c \quad (4.29)$$

we see that the first three propagators of eq. (4.28) are of the form

$$s_{cd} \times \left( f(\alpha, \beta, |\gamma|) - \frac{\mu^2}{s_{cd}} \right), \quad (4.30)$$



while the final one also contains

$$-2B \cdot x = \alpha s_{cx} + (\beta - 1)s_{dx} + \gamma [xc] \langle dx \rangle + \bar{\gamma} [xd] \langle cx \rangle = \left( \alpha + \gamma' \sqrt{\frac{s_{dx}}{s_{cx}}} \right) s_{cx} + \left( \beta - 1 + \sqrt{\frac{s_{cx}}{s_{dx}}} \bar{\gamma}' \right) s_{dx}, \quad (4.31)$$

where we have absorbed a pure phase into  $\gamma'$ . We thus have,

$$(B - x)^2 - \mu^2 = s_{cx} \left( F\left(\alpha, \beta, \gamma', \bar{\gamma}', \frac{s_{dx}}{s_{cx}}, \frac{s_{cd}}{s_{cx}}\right) - \frac{\mu^2}{s_{cx}} \right) \quad (4.32)$$

The first three propagators of eq. (4.28) combine with the Jacobian to give the required  $\langle cd \rangle^{-1}$  pole while the fourth must introduce a spurious pole: the rational term does not involve  $\mu$ , so we have an undetermined function depending on  $s_{dx}/s_{cx}$  and  $s_{cd}/s_{cx}$ . As written we have introduced a spurious  $[cx]$  factor in the denominator, while this could be cancelled by either a  $s_{cx}/s_{dx}$  or  $s_{cx}/s_{cd}$  factor, either of these would introduce a different spurious factor ( $[dx]$  or  $\langle cd \rangle$  respectively). The rational term can therefore be no more than what is required to cancel the spurious poles generated by the cut-constructible pieces of the integral.

As we are interested in the  $\langle cd \rangle \rightarrow 0$  limit, there are two cuts we might consider:  $\{d, x\}\{\bar{x}, e, c\}$  and  $\{\bar{x}, e\}\{c, d, x\}$ . On spurious poles the coefficients of pairs of bubbles coincide (up to a sign) so that the logarithms in the integral functions cancel leaving a rational descendant. Therefore we only need to calculate one bubble coefficient, with the  $\{d, x\}\{\bar{x}, e, c\}$  cut being the most natural. A direct parametrization yields a purely rational cut integral, so there are no box or triangle contributions to this cut and to leading order in  $\langle cd \rangle$  the bubble coefficient generated by the bare integral is

$$c_{x, \bar{x}; B_5}^{\text{bub, bare}} = \frac{1}{\langle cd \rangle} \left( \frac{[cd]}{[xc]} \right)^2 \frac{1}{2} \langle ad \rangle^2 \langle a|P|x \rangle \quad (4.33)$$

Here the  $[xc]^{-2}$  factor is a remnant of the spurious singularity  $[c|d+x|c\rangle$  in the  $\langle cd \rangle \rightarrow 0$  limit. Thanks to the  $[C|A|x\rangle$  factor in the numerator the order of this spurious pole is one less than the loop momentum power count.

The rational descendants of the bubbles are obtained by multiplying the bubble coefficient by the expansion of the difference of the two integral functions:

$$I_2[s_{dx}] - I_2[s_{e\bar{x}}] = \log\left(\frac{s_{e\bar{x}}}{s_{dx}}\right) = \log\left(\frac{s_{cx} + s_{dx}}{s_{dx}}\right) + \mathcal{O}(\langle cd \rangle) = \frac{s_{cx}}{s_{dx}} - \frac{1}{2} \frac{s_{cx}^2}{s_{dx}^2} + \frac{1}{3} \frac{s_{cx}^3}{s_{dx}^3} + \dots \quad (4.34)$$

A term involving  $[xc]^{-1}$  thus produces a finite rational descendant which need not be cancelled by the rational term. However terms with higher order poles would contribute to a spurious singularity and so must be cancelled. We can read off the appropriate rational term by taking all the terms from the expansion that leave at least one factor of  $1/[xc]$  uncanceled. So that this procedure does not introduce other spurious poles we must utilize any factors of  $[cx]$  or  $[dx]$  present in the numerator of the bare bubble coefficient (in particular for those bare bubble coefficients containing  $[ex]$  factors we use  $[cd][ex] = [ed][cx] + [ce][dx]$ ). To leading order we can also exploit the  $\langle ad \rangle^n$  factor using:

$$\frac{s_{cx}}{s_{dx}} = \frac{[cx]\langle xc \rangle \langle da \rangle}{[dx]\langle xd \rangle \langle da \rangle} = \frac{[cx]\langle xd \rangle \langle ca \rangle}{[dx]\langle xd \rangle \langle da \rangle} + \mathcal{O}(\langle cd \rangle) = \frac{[cx]\langle ca \rangle}{[dx]\langle da \rangle} + \mathcal{O}(\langle cd \rangle) \quad (4.35)$$

For the  $B_5$  contribution we note that

$$c_{x,\bar{x}:B_5}^{\text{bub,bare}} = \frac{1}{\langle cd \rangle} \left( \frac{[cd]}{[xc]} \right)^2 \frac{1}{2} \langle ad \rangle^2 \langle a|d|x \rangle + \mathcal{O}([xc]^{-1}) \quad (4.36)$$

and obtain the rational term,

$$\mathcal{R}_{x,\bar{x}:B_5}^{\text{bare}}(a, b, c, d, e) = \left( \frac{1}{\langle cd \rangle} \left( \frac{[cd]}{[xc]} \right)^2 \frac{1}{2} \langle ad \rangle^2 \langle a|d|x \rangle \right) \frac{[cx] \langle ca \rangle}{[dx] \langle da \rangle} = \frac{1}{2} \frac{[cd]^2 \langle da \rangle^2 \langle ca \rangle}{\langle cd \rangle [xc]} \quad (4.37)$$

The rational pieces of the other box integrals can be obtained using the same procedure, giving the rational term

$$\begin{aligned} \mathcal{R}(a, b, c, d, e) = & \mathcal{R}_{a,a:A}(a, b, c, d, e) + \mathcal{R}_{b,b:A}(a, b, c, d, e) + \mathcal{R}_{a,a:B}(a, b, c, d, e) + \mathcal{R}_{b,b:B}(a, b, c, d, e) \\ & + \sum_{i=1}^5 \left( \mathcal{R}_{a,b:A_i}(a, b, c, d, e) + \mathcal{R}_{b,a:A_i}(a, b, c, d, e) \right) \\ & + \sum_{i=1}^5 \left( \mathcal{R}_{a,b:B_i}(a, b, c, d, e) + \mathcal{R}_{b,a:B_i}(a, b, c, d, e) \right) \end{aligned} \quad (4.38)$$

where  $\mathcal{R}_{x,\bar{x}:A_i}$  denotes the full rational term generated by the box integral  $I_{x,\bar{x}:A_i}$  (i.e. the bare integral multiplied by its pre-factor). There are only simple poles in these contributions so we find

$$R_5^{ns} = \frac{1}{\langle 34 \rangle} \left( \mathcal{R}(\hat{1}, 2, \hat{3}, 4, 5) \langle \hat{3}4 \rangle \right) \Big|_{\langle \hat{3}4 \rangle \rightarrow 0} + \frac{1}{\langle 35 \rangle} \left( \mathcal{R}(\hat{1}, 2, \hat{3}, 5, 4) \langle \hat{3}5 \rangle \right) \Big|_{\langle \hat{3}5 \rangle \rightarrow 0} \quad (4.39)$$

## B. Box Contribution

There is only one type of box at five-point and their coefficients are readily evaluated using quadruple cuts [8]:

$$c_{\text{box}}^{\text{scalar}}(c^+, a^-, d^+, \{b^-, e^+\}) = \frac{(-1)^5}{2} \frac{\langle da \rangle^4 \langle ca \rangle^4 [ac]^2 [ad]^2 [be] \langle bc \rangle^3 \langle bd \rangle^3}{\langle cd \rangle^8 \langle be \rangle \langle ce \rangle \langle de \rangle}. \quad (4.40)$$

Around the  $\langle cd \rangle = 0$  pole the truncated one-mass box integral function has the expansion,

$$I_{4:\text{trunc}}^{1m}(s, t, u, m) = u(f_s \log(s) + f_t \log(t) + f_m \log(m)) + u^2 f_r, \quad (4.41)$$

where

$$f_x = \sum_{j=0}^{\infty} u^j f_{xj}(s, t, m), \quad (4.42)$$

the  $f_{xj}$  are rational functions and for this box  $u = s_{cd}$ ,  $s = s_{ac}$ ,  $t = s_{ad}$  and  $m = s_{be}$ . Given the  $\langle cd \rangle^{-8}$  singularity in the box coefficient, on the  $\langle cd \rangle = 0$  pole the box contributions produce logarithmic and rational descendants with leading singularities  $\langle cd \rangle^{-7}$  and  $\langle cd \rangle^{-6}$  respectively. The logarithmic descendants combine with the bubble contributions to leave  $\langle cd \rangle^{-1}$  singularities in the effective coefficients of the logarithms. The multiple poles in the

rational descendants cancel against terms in  $R_5$ .

The full rational descendant of this box contribution is

$$\mathcal{D}_{\text{box}}(a, b, c, d, e) = c_{\text{box}}^{\text{scalar}}(c^+, a^-, d^+, \{b^-, e^+\}) \times \mathcal{R}^7(s_{ac}, s_{ad}, s_{cd}, s_{be}), \quad (4.43)$$

where, since  $c_{\text{box}}^{\text{scalar}}$  has poles of order eight,  $\mathcal{R}^7$  is the expansion of  $u^2 f_r$  truncated to seventh order in  $u$ ,

$$\begin{aligned} \mathcal{R}^7(s, t, u, m) = & \frac{u^2}{s^2 t^2} - \frac{1}{3} \frac{u^3 m}{s^3 t^3} + \frac{1}{12} \frac{u^4 (2s^2 + 5st + 2t^2)}{s^4 t^4} - \frac{1}{30} \frac{u^5 m (3s^2 - 2st + 3t^2)}{s^5 t^5} \\ & + \frac{1}{720} \frac{u^6 (48s^4 + 84s^3 t - 40s^2 t^2 + 84st^3 + 48t^4)}{s^6 t^6} - \frac{1}{210} \frac{u^7 m (10s^4 - 8s^3 t + 9s^2 t^2 - 8st^3 + 10t^4)}{s^7 t^7} \end{aligned} \quad (4.44)$$

The shift *excites* the  $\langle cd \rangle = 0$  type pole in four of the six box contributions, leading to

$$\begin{aligned} R_5^{\text{box}} = & \text{Res}_{\langle \hat{3}4 \rangle = 0} \left( \frac{1}{z} \left( \mathcal{D}_{\text{box}}(\hat{1}, 2, \hat{3}, 4, 5) + \mathcal{D}_{\text{box}}(2, \hat{1}, \hat{3}, 4, 5) \right) \right) \\ & + \text{Res}_{\langle \hat{3}5 \rangle = 0} \left( \frac{1}{z} \left( \mathcal{D}_{\text{box}}(\hat{1}, 2, \hat{3}, 5, 4) + \mathcal{D}_{\text{box}}(2, \hat{1}, \hat{3}, 5, 4) \right) \right) \end{aligned} \quad (4.45)$$

### C. Bubble Contribution

The bubble coefficients can be obtained from two particle cuts using canonical forms [10]. There is a single type of bubble with coefficient  $c_{\text{bub}}^{\text{scalar}}(\{a^-, c^+\}, \{b^-, d^+, e^+\})$ . The general form of the bubble coefficients is given in appendix B.

The bubble coefficients contain poles of the form  $\langle cd \rangle^{-7}$ . On the  $\langle cd \rangle \rightarrow 0$  poles these terms are precisely those required to cancel the multiple poles in the coefficients of the logarithmic descendants of the boxes. The bubble coefficients also contain spurious poles of the form  $[d|a + c|d]^{-5}$  (two powers worse than the  $\mathcal{N} = 1$  case). On these poles pairs of bubble contributions combine to produce rational descendants with  $[d|a + c|d]^{-4}$  spurious poles. The singular pieces of these rational descendants are

$$\mathcal{D}_{\text{bub}}(1, 2, 3, 4, 5) = \left( \mathcal{S}(1, 2, 3, 4, 5) + \mathcal{P}_{\{3,4,5\}} \right) + \left( \mathcal{S}(2, 1, 3, 4, 5) + \mathcal{P}_{\{3,4,5\}} \right) \quad (4.46)$$

where  $\mathcal{P}_{\{3,4,5\}}$  represents permutations of the positive helicity legs and

$$\begin{aligned}
\mathcal{S}(a, b, c, d, e) = & \frac{\langle a c \rangle^4 \langle b d \rangle^6 [b e] s_{cd}^3}{60 \langle b e \rangle \langle c d \rangle^8 \langle d e \rangle^2 [d|a+c|d]} \\
\times & \left( \frac{s_{cd}^3}{[d|a+c|d]^3} - \frac{1}{2} \frac{s_{cd}^2}{[d|a+c|d]^2} \left[ \frac{s_{cd}}{s_{ac}} + 3 \left( 1 + 3 \frac{\langle c b \rangle \langle a d \rangle}{\langle a c \rangle \langle b d \rangle} - \frac{\langle c e \rangle \langle a d \rangle}{\langle c a \rangle \langle d e \rangle} \right) \right] \right. \\
& + \frac{1}{12} \frac{s_{cd}}{[d|a+c|d]} \left[ 4 \frac{s_{cd}^2}{s_{ac}^2} + 9 \frac{s_{cd}}{s_{ac}} \left( 1 + 3 \frac{\langle c b \rangle \langle a d \rangle}{\langle a c \rangle \langle b d \rangle} - \frac{\langle c e \rangle \langle a d \rangle}{\langle c a \rangle \langle d e \rangle} \right) \right. \\
& \left. \left. + 30 \left( 3 \frac{\langle c b \rangle^2 \langle a d \rangle^2}{\langle a c \rangle^2 \langle b d \rangle^2} - 3 \frac{\langle c b \rangle \langle a d \rangle \langle c e \rangle \langle a d \rangle}{\langle a c \rangle \langle b d \rangle \langle c a \rangle \langle d e \rangle} + \frac{\langle c e \rangle^2 \langle a d \rangle^2}{\langle c a \rangle^2 \langle d e \rangle^2} \right) \right] \right. \\
& + \frac{1}{2} \frac{s_{cd}^2}{s_{ac}^2} \left( -1 - 3 \frac{\langle c b \rangle \langle a d \rangle}{\langle a c \rangle \langle b d \rangle} \right) - \frac{15}{4} \frac{s_{cd}}{s_{ac}} \frac{\langle a d \rangle^2 \langle c b \rangle \langle c d \rangle \langle b e \rangle}{\langle a c \rangle^2 \langle b d \rangle^2 \langle d e \rangle} + 10 \frac{\langle c e \rangle \langle a d \rangle}{\langle a c \rangle \langle d e \rangle} \\
& \left. + 5 \left[ \frac{\langle c d \rangle^2}{\langle a c \rangle^2} \left( 3 \frac{\langle a b \rangle^2}{\langle b d \rangle^2} + 4 \frac{\langle a e \rangle^2}{\langle d e \rangle^2} + 9 \frac{\langle a b \rangle \langle a e \rangle}{\langle b d \rangle \langle d e \rangle} \right) - \frac{\langle c d \rangle^3 \langle a b \rangle^2}{\langle a c \rangle^3 \langle b d \rangle^2} \left( \frac{\langle a b \rangle}{\langle b d \rangle} + 3 \frac{\langle a e \rangle}{\langle d e \rangle} \right) \right] \right) \quad (4.47)
\end{aligned}$$

The shift excites four distinct spurious poles:  $[\hat{3}|\hat{1}+4|\hat{3}\rangle = 0$ ,  $[\hat{3}|\hat{1}+5|\hat{3}\rangle = 0$ ,  $[4|\hat{1}+5|4\rangle = 0$  and  $[5|\hat{1}+4|5\rangle = 0$  and we have

$$\begin{aligned}
R_5^{\text{bub}}(1, 2, 3, 4, 5) = & \text{Res}_{[\hat{3}|\hat{1}+4|\hat{3}\rangle=0} \left( \frac{1}{z} \mathcal{D}_{\text{bub}}(\hat{1}, 2, \hat{3}, 4, 5) \right) + \text{Res}_{[\hat{3}|\hat{1}+5|\hat{3}\rangle=0} \left( \frac{1}{z} \mathcal{D}_{\text{bub}}(\hat{1}, 2, \hat{3}, 4, 5) \right) \\
& + \text{Res}_{[4|\hat{1}+5|4\rangle=0} \left( \frac{1}{z} \mathcal{D}_{\text{bub}}(\hat{1}, 2, \hat{3}, 4, 5) \right) + \text{Res}_{[5|\hat{1}+4|5\rangle=0} \left( \frac{1}{z} \mathcal{D}_{\text{bub}}(\hat{1}, 2, \hat{3}, 4, 5) \right) \quad (4.48)
\end{aligned}$$

The residues of all terms are readily extracted using **Mathematica**. This completes the computation of  $R_5$ . Technically, the validity of the above result relies upon, *a priori*, the vanishing of  $R(z)$  for large  $z$  for which no general theorems are available. Our expression satisfies several non-trivial consistency conditions which, experience suggests, make it almost certainly correct. In particular, the resulting expression for the amplitude has the correct symmetries, is free from spurious poles, has the correct soft and collinear limits and has the correct complex factorisation. The **Mathematica** expression for  $R_5$  is available at <http://pyweb.swan.ac.uk/~dunbar/graviton.html>.

Our results are consistent with the suggestion that gravity perturbation theory has a significantly softer ultra-violet behaviour at one-loop than traditionally expected [26, 32, 33]. If a Feynman diagram has  $n$ -points in the loop and has a loop momentum polynomial of degree  $m$  then bubble coefficients have  $(a \cdot P)$  spurious singularities of the form

$$\sim \frac{1}{(a \cdot P)^{m-n+3}} \quad (4.49)$$

and the rational terms have one power less. For gravity the traditional expectation of  $m = 2n$  leads to  $(a \cdot P)^{-n-2}$ . However, our explicit calculations indicate a softer behaviour. The bubble coefficients (B11) have explicit  $(a \cdot P)^{-6}$  singularities arising from the  $H_{2x}^4$  terms. For the five-point case the leading singularity vanishes leaving  $(a \cdot P)^{-5}$  singularities - and consequently only  $(a \cdot P)^{-4}$  singularities in the rational term. These singularities are thus consistent with

an effective power count of  $m = n + 4$ . This is in agreement with the expectations of [26, 33].

## V. CONCLUSIONS

Recursive techniques based on the factorisation of rational tree amplitudes have proved of great utility. In extending these methods to one-loop amplitudes one faces obstacles of various types: in general one-loop amplitudes contain both rational and transcendental functions along with a plethora of spurious poles and higher order physical poles. The transcendental pieces of one-loop amplitudes are readily obtained using four dimensional unitarity techniques. When the coefficients of these transcendental functions contain high order poles these contributions give rational descendants which must be accounted for when computing the remaining rational terms recursively. Further, using recursive techniques for the computation of one-loop amplitudes requires an understanding of the singularities of these amplitudes when extended to complex momenta. For complex momenta there are “non-standard” factorisations which must be accounted for.

In this article we have demonstrated how axial gauge techniques may be used to determine the non-standard factorisations. Axial gauge techniques provide a natural method for examining complex structures since they preserve the language and structure of on-shell amplitudes, however, like usual Feynman diagram techniques, they can prove rather cumbersome. Nonetheless, we have used these techniques to A) prove the conjectured sub-leading pole in the single-minus Yang-Mills amplitude, B) compute the rational terms of the five graviton MHV amplitude analytically. This completes the calculation of the five graviton scattering amplitude. The expression we obtain is not particularly simple but will provide a significant target for alternative techniques. Our results are consistent with the suggestion that gravity perturbation theory has a significantly softer UV behaviour than traditionally expected.

The work of S. Alston was supported by a STFC studentship.

## Appendix A: Box Integrals Contributing To The Non-standard Factorisations

The non-standard factorisation term  $I_{x,\bar{x}}^{[0]}$  can be expressed as a sum of box integrals:

$$I_{x,\bar{x}}^{[0]} = \sum_{i=1}^5 (I_{x,\bar{x}:A_i}^{[0]} + I_{x,\bar{x}:B_i}^{[0]}) + I_{x,\bar{x}:P}^{[0]} + \mathcal{O}(\langle cd \rangle). \quad (\text{A1})$$

Using the  $\ell$ -dependent variables  $A = \ell + c$ ,  $B = -\ell + d$ , the integrals required are

$$I_{x,\bar{x}:A_1}^{[0]} = \frac{[e\bar{x}]}{[ab]^2 \langle ca \rangle^2 \langle da \rangle^2 [\bar{x}x]} \int \frac{d^d \ell}{\ell^2 A^2 B^2} \frac{[c|A|a][d|B|a][e|B|a][e|A|a][c|A|\bar{x}][d|B|x][e|A|\bar{x}]}{(A + \bar{x})^2}, \quad (\text{A2})$$

$$I_{x,\bar{x}:B_1}^{[0]} = \frac{[ex]}{[ab]^2 \langle ca \rangle^2 \langle da \rangle^2 [\bar{x}x]} \int \frac{d^d \ell}{\ell^2 A^2 B^2} \frac{[c|A|a][d|B|a][e|B|a][e|A|a][c|A|\bar{x}][d|B|x][e|B|x]}{(B - x)^2}, \quad (\text{A3})$$

$$I_{x,\bar{x}:A_2}^{[0]} = \frac{[e\bar{x}][xe][c\bar{x}]}{[ab]^4 \langle ca \rangle^2 \langle da \rangle^2} \int \frac{d^d \ell}{\ell^2 A^2 B^2} \frac{[c|A|a][d|B|a][e|B|a][e|A|a]\langle \bar{x}|AB|x\rangle[d|A|\bar{x}]}{(A + \bar{x})^2}, \quad (\text{A4})$$

$$I_{x,\bar{x}:B_2}^{[0]} = \frac{[e\bar{x}][xe][cx]}{[ab]^4\langle ca\rangle^2\langle da\rangle^2} \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{[c|A|a\rangle[d|B|a\rangle[e|B|a\rangle[e|A|a\rangle\langle\bar{x}|AB|x\rangle[d|B|x\rangle]}{(B-x)^2}, \quad (\text{A5})$$

$$I_{x,\bar{x}:A_3}^{[0]} = \frac{[e\bar{x}]^2[xe][c\bar{x}][xd]}{[ab]^4\langle ca\rangle^2\langle da\rangle^2[\bar{x}x]} \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{[c|A|\bar{x}\rangle[d|B|x\rangle[e|A|\bar{x}\rangle\langle a|AB|a\rangle^2]}{(A+\bar{x})^2}, \quad (\text{A6})$$

$$I_{x,\bar{x}:B_3}^{[0]} = \frac{[ex]^2[\bar{x}e][c\bar{x}][xd]}{[ab]^4\langle ca\rangle^2\langle da\rangle^2[\bar{x}x]} \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{[c|A|\bar{x}\rangle[d|B|x\rangle[e|B|x\rangle\langle a|AB|a\rangle^2]}{(B-x)^2}, \quad (\text{A7})$$

$$I_{x,\bar{x}:A_4}^{[0]} = \frac{[ex]^2[e\bar{x}]^2[c\bar{x}][xd][\bar{x}|P|x\rangle]}{[ab]^6\langle ca\rangle^2\langle da\rangle^2} \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{[c|A|a\rangle[d|B|a\rangle\langle\bar{x}|AP|\bar{x}\rangle\langle a|AB|a\rangle]}{(A+\bar{x})^2}, \quad (\text{A8})$$

$$I_{x,\bar{x}:B_4}^{[0]} = \frac{[ex]^2[e\bar{x}]^2[c\bar{x}][xd][x|P|x\rangle]}{[ab]^6\langle ca\rangle^2\langle da\rangle^2} \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{[c|A|a\rangle[d|B|a\rangle\langle x|BP|\bar{x}\rangle\langle a|AB|a\rangle]}{(B-x)^2}, \quad (\text{A9})$$

$$I_{x,\bar{x}:A_5}^{[0]} = \frac{[ex]^2[e\bar{x}]^2[c\bar{x}][xd][\bar{x}|P|x\rangle[x|P|\bar{x}\rangle[\bar{x}|P|a\rangle]}{[ab]^6\langle ca\rangle^2\langle da\rangle^2[\bar{x}x]} \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{[c|A|a\rangle[d|B|a\rangle\langle\bar{x}|AP|a\rangle]}{(A+\bar{x})^2}, \quad (\text{A10})$$

$$I_{x,\bar{x}:B_5}^{[0]} = \frac{[ex]^2[e\bar{x}]^2[c\bar{x}][xd][\bar{x}|P|x\rangle[x|P|\bar{x}\rangle[x|P|a\rangle]}{[ab]^6\langle ca\rangle^2\langle da\rangle^2[\bar{x}x]} \int \frac{d^d\ell}{\ell^2 A^2 B^2} \frac{[c|A|a\rangle[d|B|a\rangle\langle x|BP|a\rangle]}{(B-x)^2}. \quad (\text{A11})$$

## Appendix B: Bubble Coefficients for $n$ -point MHV amplitudes

In this appendix we present a general formula for the bubble coefficients of the  $n$ -graviton MHV scattering amplitude. The bubble coefficients vanish for  $\mathcal{N} = 8, 6$  contributions. The  $\mathcal{N} = 4$  contributions were given in ref. [26] and the  $\mathcal{N} = 1$  coefficients in ref. [27]. We present the remaining scalar contribution together with these in a unified way using canonical forms [10]. In the method of canonical forms the bubble coefficients are obtained from unitarity by decomposing the product of tree amplitudes appearing in a two-particle cut into canonical forms  $\mathcal{F}_i$ ,

$$\sum M^{\text{tree}}(-\ell_1, \dots, \ell_2) \times M^{\text{tree}}(-\ell_2, \dots, \ell_1) = \sum_i c_i \mathcal{F}_i(\ell_j), \quad (\text{B1})$$

where the  $c_i$  are coefficients independent of  $\ell_j$ . The momentum across the cut is  $P = \ell_1 - \ell_2$ . We then use substitution rules to replace the  $\mathcal{F}_i(\ell_j)$  by the canonical forms  $F_i(P)$  to obtain the coefficient of the bubble integral  $I_2(P^2)$  as

$$\sum_i c_i F_i(P) \quad (\text{B2})$$

The general types of canonical forms we need for MHV amplitudes are

$$\mathcal{H}_0^n \equiv \prod_{i=1}^n [D_i | \ell_1 | B_i \rangle \quad (\text{B3})$$

$$\mathcal{H}_1^n \equiv \prod_{i=1}^n [D_i | \ell_1 | B_i \rangle \frac{\langle \ell_1 B_{n+1} \rangle}{\langle \ell_1 A \rangle} \quad (\text{B4})$$

$$\mathcal{H}_{1,1}^n \equiv \prod_{i=1}^n [D_i | \ell_1 | B_i \rangle \frac{\langle \ell_1 B_{n+1} \rangle}{\langle \ell_1 A_1 \rangle} \frac{\langle \ell_2 C_1 \rangle}{\langle \ell_2 A_2 \rangle} \quad (\text{B5})$$

$$\mathcal{H}_{2x}^n \equiv \prod_{i=1}^n [D_i | \ell_1 | B_i \rangle \frac{\langle \ell_1 B_{n+1} \rangle}{\langle \ell_1 A \rangle} \frac{\langle \ell_2 C_1 \rangle}{\langle \ell_2 A \rangle} \quad (\text{B6})$$

Supersymmetric amplitudes require canonical forms with lower values of  $n$ . In Yang-Mills theory we need values  $n \leq 2$  in general but only  $n = 0$  for supersymmetric contributions. For gravity, scalar contributions require canonical forms up to  $n = 4$  with only  $n \leq 2$  for  $\mathcal{N} = 1$  and  $n = 0$  for  $\mathcal{N} = 4$ . The expressions for the canonical forms with  $n > 2$  were not given previously.

We first note some general results (not given previously)

$$\begin{aligned} H_0^n[\{B_i\}; \{D_i\}] &= \frac{1}{(n+1)!} \sum_{P(B_i)} \prod_{i=1}^n [D_i | P | B_i \rangle \\ H_1^n[A; \{B_i\}; \{D_i\}] &= \frac{1}{((n+1)!)^2} \sum_{P(B_i), P(D_i)} \sum_{r=0}^n \frac{(P^2)^{n-r}}{[A|P|A]^{n-r+1}} \prod_{i=1}^r [D_i | P | B_i \rangle \prod_{j=1}^{n-r} [D_i | A | B_j \rangle [A | P | B_{n+1} \rangle \end{aligned} \quad (\text{B7})$$

$H_{2x}^1$  and  $H_{2x}^2$  are given in ref [10]. The expressions for  $H_{2x}^3$  and  $H_{2x}^4$  are

$$\begin{aligned}
H_{2x}^3[A; \{B_1, B_2, B_3, B_4\}; C_1; \{D_1, D_2, D_3\}; P] = & \frac{1}{576} \sum_{P(\{B_i\}), P(\{D_i\})} \left( \right. \\
& \frac{(P^2)^3}{[A|P|A]^5} \left( 4[D_1|A|B_1][D_2|A|B_2][D_3|A|B_3][A|P|B_4][A|P|C_1] \right. \\
& \quad \left. - 4[D_1|A|B_1][D_2|A|B_2][D_3|A|C_1][A|P|B_3][A|P|B_4] \right) \\
& + \frac{(P^2)^2}{[A|P|A]^4} \left( 3[D_1|A|B_1][D_2|A|B_2][D_3|P|C_1][A|P|B_3][A|P|B_4] \right) \\
& + \frac{(P^2)}{[A|P|A]^3} \left( 2[D_1|A|B_1][D_2|P|B_2][D_3|P|C_1][A|P|B_3][A|P|B_4] \right. \\
& \quad \left. - 4[D_1|A|C_1][D_2|P|B_1][D_3|P|B_2][A|P|B_3][A|P|B_4] \right) \\
& \left. + \frac{1}{[A|P|A]^2} \left( [D_1|P|B_1][D_2|P|B_2][D_3|P|C_1][A|P|B_3][A|P|B_4] \right) \right) \quad (B8)
\end{aligned}$$



and

$$\begin{aligned}
H_{2x}^4[A; \{B_1, B_2, B_3, B_4, B_5\}; C_1; \{D_1, D_2, D_3, D_4\}; P] &= \frac{1}{5!6!} \sum_{P(\{B_i\}), P(\{D_i\})} \left( \right. \\
&\frac{(P^2)^4}{[A|P|A]^6} \left( 12[D_1|A|B_1\rangle[D_2|A|B_2\rangle[D_3|A|B_3\rangle[D_4|A|B_4\rangle[A|P|B_5\rangle[A|P|C_1\rangle \right. \\
&\quad \left. - 2[D_1|A|B_1\rangle[D_2|A|B_2\rangle[D_3|A|B_3\rangle[D_4|A|C_1\rangle[A|P|B_4\rangle[A|P|B_5\rangle] \right. \\
&+ \frac{(P^2)^3}{[A|P|A]^5} \left( 6[D_1|A|B_1\rangle[D_2|A|B_2\rangle[D_3|A|B_3\rangle[D_4|P|C_1\rangle[A|P|B_4\rangle[A|P|B_5\rangle \right. \\
&\quad \left. - 22[D_1|A|B_1\rangle[D_2|A|B_2\rangle[D_3|A|C_1\rangle[D_4|P|B_3\rangle[A|P|B_4\rangle[A|P|B_5\rangle] \right. \\
&+ \frac{(P^2)^2}{[A|P|A]^4} \left( 33[D_1|A|B_1\rangle[D_2|A|B_2\rangle[D_3|P|B_3\rangle[D_4|P|B_4\rangle[A|P|B_5\rangle[A|P|C_1\rangle] \right. \\
&+ \frac{(P^2)}{[A|P|A]^3} \left( 27[D_1|A|B_1\rangle[D_2|P|B_2\rangle[D_3|P|B_3\rangle[D_4|P|C_1\rangle[A|P|B_4\rangle[A|P|B_5\rangle \right. \\
&\quad \left. - 45[D_1|A|C_1\rangle[D_2|P|B_1\rangle[D_3|P|B_2\rangle[D_4|P|B_3\rangle[A|P|B_4\rangle[A|P|B_5\rangle] \right. \\
&+ \frac{1}{[A|P|A]^2} \left( -15[D_1|P|B_1\rangle[D_2|P|B_2\rangle[D_3|P|B_3\rangle[D_4|P|B_4\rangle[A|P|B_5\rangle[A|P|C_1\rangle \right. \\
&\quad \left. + 21[D_1|P|B_1\rangle[D_2|P|B_2\rangle[D_3|P|B_3\rangle[D_4|P|C_1\rangle[A|P|B_4\rangle[A|P|B_5\rangle] \right. \left. \right) \left. \right) \quad (B9)
\end{aligned}$$

We do not need to present  $H_{1,1}^n$  separately since

$$\begin{aligned}
H_{1,1}^n[A_1; A_2; B_1, \dots, B_{n+1}; C_1; D_1, \dots, D_n; P] &= H_2^n[A_1, A_2; B_1, \dots, B_{n+1}, C_1; D_1, \dots, D_n; P] \\
&+ \frac{\langle C_1 A_2 \rangle \langle A_2 B_1 \rangle}{\langle A_2 A_1 \rangle} H_{2x}^{n-1}[A_2; B_2, \dots, B_n, B_{n+1}; P|D_1]; D_2, \dots, D_n; P] \\
&- \frac{\langle C_1 A_2 \rangle \langle B_1 A_1 \rangle}{\langle A_2 A_1 \rangle} H_{1,1}^{n-1}[A_1; A_2; B_2, \dots, B_n, B_{n+1}; P|D_1]; D_2, \dots, D_n; P] \quad (B10)
\end{aligned}$$

We can now use these canonical forms to provide expressions for the coefficients of the bubble integrals  $I_2(P^2)$ . The bubble integral functions  $I_2(P^2)$  will have vanishing coefficients for the MHV amplitude unless the momentum  $P$  (and hence  $-P$ ) contains exactly one negative helicity leg and at least one positive helicity leg. We can thus take  $P$  to be of the form  $\{m_1^-, a_1^+, a_2^+, \dots, a_{n_L}^+\}$  and the legs on the other side of the cut to be  $\{m_2^-, b_1^+, b_2^+, \dots, b_{n_R}^+\}$ .

Using the canonical forms we find the bubble coefficient

$$\begin{aligned}
c(m_1, \{a_i\}; m_2, \{b_j\}) &= \frac{\langle m_1 m_2 \rangle^M}{(P^2)^{4-M}} \sum_{P_L, P_R} C_{P_L} C_{P_R} \left( \right. \\
&\sum_{x \in \{a_i\} \cup \{b_j\} - a_1} D_x \frac{\langle m_2 a_1 \rangle}{\langle b_1 a_1 \rangle} H_{1,1}^{4-M}[a_1; x; \{B_i\}; m_1; \{D_i\}; P] \\
&+ \sum_{x \in \{a_i\} \cup \{b_j\} - b_1} D_x \frac{\langle m_2 b_1 \rangle}{\langle a_1 b_1 \rangle} H_{1,1}^{4-M}[b_1; x; \{B_i\}; m_1; \{D_i\}; P] \\
&\left. + D_{a_1} \frac{\langle m_2 a_1 \rangle}{\langle b_1 a_1 \rangle} H_{2x}^{4-M}[a_1; \{B_i\}; m_1; \{D_i\}; P] + D_{b_1} \frac{\langle m_2 b_1 \rangle}{\langle a_1 b_1 \rangle} H_{2x}^{4-M}[b_1; \{B_i\}; m_1; \{D_i\}; P] \right) \quad (\text{B11})
\end{aligned}$$

where  $P_L$  and  $P_R$  are permutations of the positive helicity legs  $\{a_i\}$  and  $\{b_i\}$  respectively. The different cases are specified by  $M = 2$  for the  $\mathcal{N} = 4$  matter multiplet,  $M = 1$  for the  $\mathcal{N} = 1$  matter multiplet and  $M = 0$  for the scalar contribution. The arguments are

$$\begin{aligned}
\{B_i\}^{\mathcal{N}=4} &= \{m_1\}, \{B_i\}^{\mathcal{N}=1} = \{m_1, m_2, m_1\}, \{B_i\}^{[0]} = \{m_1, m_2, m_1, m_2, m_1\} \\
\{D_i\}^{\mathcal{N}=4} &= \emptyset, \{D_i\}^{\mathcal{N}=1} = \{P|m_1\rangle, P|m_2\rangle\}, \{D_i\}^{[0]} = \{P|m_1\rangle, P|m_2\rangle, P|m_1\rangle, P|m_2\rangle\} \quad (\text{B12})
\end{aligned}$$

and

$$C_{P_L} = \frac{1}{\langle n_L m_1 \rangle \prod_{i=1}^{n_L-1} \langle a_i a_{i+1} \rangle}, \quad C_{P_R} = \frac{1}{\langle n_R m_2 \rangle \prod_{j=1}^{n_R-1} \langle b_j b_{j+1} \rangle}, \quad (\text{B13})$$

$$D_x = \frac{\langle m_2 x \rangle \prod_{l=1}^{n_L-1} [a_l | \tilde{K}_{l+1} | x] \prod_{k=1}^{n_R-1} [b_k | \tilde{K}'_{k+1} | x]}{\prod_{y \neq x} \langle x y \rangle} = \frac{\prod_{l=1}^{n_L-1} [a_l | \tilde{K}_{l+1} | x] \prod_{k=1}^{n_R} [b_k | \tilde{K}'_{k+1} | x]}{[b_{n_R} m_2] \prod_{y \neq x} \langle x y \rangle} \quad (\text{B14})$$

where  $\tilde{K}_p = k_{a_p} + \dots k_{a_{n_L}} + k_{m_1}$  and  $\tilde{K}'_p = k_{b_p} + \dots k_{b_{n_R}} + k_{m_2}$ . The bubble coefficients satisfy, nontrivially, the IR relation  $\sum_i c_i = 0$  [34].

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